Expressing entropy globally in terms of (4D) field-correlations⋆

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Abstract

We express the entropy of a scalar field ϕ directly in terms of its spacetime correlation function $W(x,y) = \langle\phi(x)\phi(y)\rangle$, assuming that the higher correlators are of "Gaussian" form. The resulting formula associates an entropy $S(R)$ to any spacetime region R; and when R is globally hyperbolic with Cauchy surface Σ, $S(R)$ can be interpreted as the entropy of the reduced density-matrix belonging to Σ. One acquires in particular a new expression for the entropy of entanglement across an event-horizon. Thanks to its spacetime character, this expression makes sense in a causal set as well as in a continuum spacetime.

As usually conceived of, entropy is local in time, being defined relative to a hypersurface Σ. To compute such an entropy (at least in its "Gibbsian" guise) one must be able to identify a density-matrix $\rho(\Sigma)$ which one can plug into the formula $S = \mathrm{Tr}\,\rho\log\rho^{-1}$. But according to our current understanding, quantum fields are in general too singular to admit of meaningful restriction to lower dimensional subsets of spacetime. If it is therefore doubtful whether a concept like "state on a hypersurface" can be well defined, it is virtually certain that this concept will break down in the discrete context of a causal set. For reasons such as these it would be desirable to define entropy in a more global manner

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by associating it with a spacetime region rather than a submanifold of codimension one. A more global notion of entropy would also seem fitting in connection with black holes, whose very definition is global in character.

Of course, black hole entropy will not be understood fully except against the background of a theory of quantum gravity and quantum spacetime. Nevertheless it seems fair to say that quantum correlations between the hole's interior and its exterior must contribute to its entropy. Perhaps, when suitably understood, some such entropy of entanglement will even turn out to tell the whole story. Be that as it may, a portion of the entanglement entropy can be identified without invoking full quantum gravity, namely that portion belonging to whatever "matter" fields are present in the neighborhood of the horizon.

When we compute such an entropy in the continuum, we obtain an infinite answer, as is well known. Introducing a cutoff ℓ on the other hand, we obtain $S_{entanglement} \sim A/\ell^2$, which compares favorably with the exact formula $S_{BH} = 2\pi A/\ell_p^2$, where $\ell_p = \sqrt{8\pi G}$ is the rationalized Planck length. [1] We recognize here the familiar area law, but with a coefficient that is not easily determined because it is not "universal".

On one hand this sensitivity of S to the details of the cutoff is to be welcomed because it lets us learn something about the magnitude of ℓ and the nature of Planck-scale physics. (Although they are sometimes viewed as uninteresting, non-universal quantities are actually the most interesting if we are after the microscopic physics!)

But on the other hand, the very fact that the answer is not universal also creates a difficulty. When we try to compute $S_{entanglement}$ with finite ℓ the answer will depend on the details of how ℓ is introduced (sometimes called, rather misleadingly, "scheme dependence"). In itself, this ambiguity might not be a problem, but for the fact that it tends to call into question the covariance of S. For example in computing entanglement entropy for a Schwarzschild black hole, we might think to neglect all contributions from within a distance ℓ of the horizon, but what does this mean? Do we neglect modes within a shell $\Delta r = \ell$ where r is the Schwarzschild radial coordinate? Or should we be using instead of r, proper distance along the spacelike surface Σ whose entropy we want (proper distance to a lightlike surface being not in itself well-defined)? But then the answer will depend on how we extend Σ away from the horizon H. If for example we define Σ as a surface of constant Schwarzschild-time t then it will go through the bifurcation 2-sphere. But realistic black holes don't have such bifurcation surfaces; and even if they did, our attention when we came to consider the crucial question of entropy *increase* would have to focus on portions of the horizon which were farther to the future.

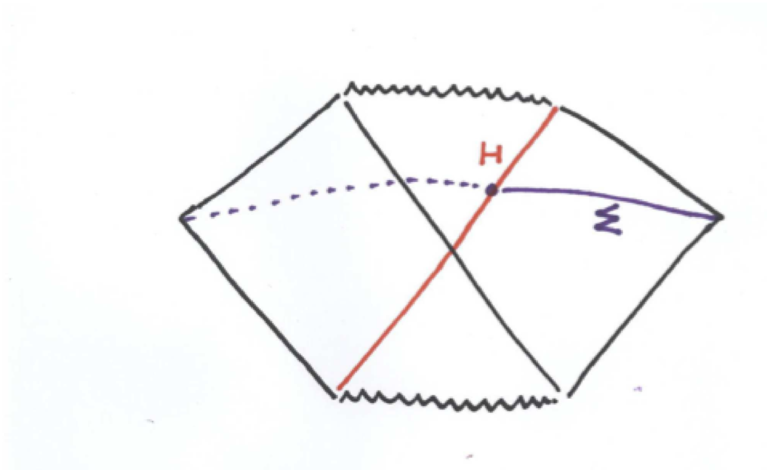

Figure 1. The black hole horizon H and a hypersurface Σ whose entropy is desired. The dotted line extends Σ to a Cauchy surface of the full spacetime.

entanglement entropy in a causal set

The difficulties we have just reviewed make it hard to define an entropy of entanglement that is at the same time covariant, but if one replaces spacetime by a causal set the difficulties disappear because no cutoff is needed. That, of course, is one reason for thinking of the causal set as more fundamental than the continuum. [2] [3] [4] But even if one is seeking no more than a "phenomenological cutoff" to render the entropy finite, a causet sprinkled into the spacetime in question offers (uniquely as far as I know) a covariant way to obtain one. For this reason alone, it is worth asking whether a natural definition of entanglement entropy can be found within causal set theory.

Let us consider then, the problem of defining a horizon-entropy in a fixed, background causal set. First of all one must attach a meaning to the words "event horizon". Although this might seem to pose a problem because a causet contains no subset that could play the role of a null surface, the distinction between the interior and exterior of a black hole still makes sense because it depends solely on the causal order. Since quantum entanglement concerns only the interior and exterior regions, and not the horizon per se, there is in fact no problem.

The next step, if we were to mimic the continuum story perfectly, would be to define the concept of a spacelike (or achronal) surface Σ , after which we would try to associate with each suitable Σ some sort of "momentary state" or density matrix ρ_Σ. In fact, there does exist a natural choice for Σ , since we can define it as an antichain, or better a maximal antichain (or "slice"). That which is not readily available in a causet, however, is anything like ρ_Σ : a causet-based field theory admits no obvious notion of "state on a hypersurface".

More generally, none of the surface-quantities which one routinely defines in the continuum seems to carry over naturally to an antichain, except possibly a measure of (spatial) volume. In fact, it seems to be a rule of thumb that "purely spatial" concepts do not live happily in a causet † including for example the concepts of induced metric g_{jk} and extrinsic curvature $K_{jk} \sim \dot{g}_{jk}$. For this reason a causet is not congenial to canonical quantum gravity, albeit one *can* define many surface-quantities in an approximate sense, including both metrical ones and topological ones like homology groups [5].

The question then is whether we can free $S_{entanglement}$ from reference to a density matrix localized to a hypersurface. In seeking the answer, I will limit myself to what is perhaps the simplest case, that of a free scalar field $\widehat{\phi}(x)$. For this case a corresponding theory exists in the causet, and it has been given both algebraic and path-integral (or "quantum measure") formulations. Inasmuch as the path-integral formulation extends to the interacting case, it is arguably preferable to the operator formulation. Nevertheless, I will work in the latter context, first of all because more tools are available there, and more

† More generally, there is evidence for the rule that it is the continuum concepts which are "C^0-stable" that have simple causet counterparts.

importantly because we seem to lack any definition of entropy couched in the language of histories.[♭]

Could it be that this lack betokens an inherent dependence of entropy on some notion of "state localized in time"? Were that true, the search for a histories-based definition of entropy would necessarily be futile, but such pessimism is called into question by the global definition of entropy we will arrive at below in the algebraic context. If indeed a more (Lorentzian and non-equilibrium) histories-based concept of entropy is out there somewhere, its discovery would, to my mind, mark important progress in the path-integral formulation of quantum theories.

Entropy from the spacetime correlator

Let us recall some possible definitions of entropy in the context of quantum field theory. Conceived algebraically (as it commonly is), a quantum field is a collection of operators[⋆] $\phi(x)$ acting irreducibly in a hilbert space $\mathfrak{H}$, together with a "global state" represented by another operator ρ (called "the density-matrix") such that $\mathrm{Tr}\,\rho A$ yields the "expectation value" $\langle A \rangle$ of A . One can then define the *entropy* as $S = S(\rho) = \mathrm{Tr}\,\rho \log \rho^{-1}$. (In practice the ρ that enters this definition will seldom be the "exact microscopic density matrix". Rather it will be some coarse-grained version thereof whose entropy is nonzero even when the underlying "microscopic state" is pure.)

Now let R be an arbitrary spacetime region. The operators $\phi(x)$ for $x \in R$ generate a subalgebra $\mathfrak{A}_R$ of $\mathfrak{A}$, and the expectation-value for $\mathfrak{A}$ restricts to a similar functional on

[♭] I'm ignoring here the trick that computes the thermodynamic partition function via a Wick-rotated path integral. That procedure is indeed histories-based, but the histories in question develop in imaginary time, and more importantly, the entropy defined thereby is limited to states of thermal equilibrium.

[⋆] In the continuum the symbol $\phi(x)$ must of course be interpreted formally since the field ϕ is at best an operator-valued distribution. In the following lines I will ignore this technicality and whatever difficulties flow from it. They are peculiar to continuous spacetime and do not seem to be germane to the causal set context of primary interest in this paper, within which the symbol ϕ^j belonging to element j of the causet will literally denote an operator, albeit an unbounded one.

$\mathfrak{A}_R$. In general, the subalgebra $\mathfrak{A}_R$ will no longer act irreducibly in $\mathfrak{H}$. Assuming however, that we can represent $\mathfrak{A}_R$ in some other Hilbert space $\mathfrak{H}_R$ in which it does act irreducibly, and assuming further that we can find in $\mathfrak{H}_R$ a density matrix ρ_R such that $\langle A \rangle = \operatorname{Tr} \rho_R A$ for operators $A \in \mathfrak{A}_R$, we can go on to define $S(R) = - \operatorname{Tr} \rho_R \log \rho_R$, *the entropy of* ρ *relative to the region* R.

REMARK Let $\mathfrak{A}$ be the algebra generated by the $\phi(x)$. As such, it is a concrete algebra of linear operators in $\mathfrak{H}$, but in what might be called the "strictly algebraic" formulations of quantum field theory, $\mathfrak{A}$ is regarded as an abstract $\star$-algebra, while a density-matrix ρ is regarded simply as an expectation-value-functional on $\mathfrak{A}$. Since the set of all such functionals is convex, any given ρ will be (either exactly or to a close approximation) a convex combination $\sum_\alpha p_\alpha \rho_\alpha$ of extremal (or "pure") states ρ_α . In this abstract setting, $S(\rho)$ could perhaps be defined as the infimum † over all such sums of the quantity $\sum_\alpha p_\alpha \log p_\alpha^{-1}$. In this way, one could imagine defining the entropy $S(R)$ directly in terms of the expectation-value functional, without recourse to either $\mathfrak{H}_R$ or ρ_R .

The definition we have just given of the entropy of an arbitrary spacetime region R will be the basis of the causal set construction we are seeking. First though, we need to relate the entropy of a region to the entropy of a hypersurface. To define the latter itself, we can imagine repeating the same steps that led to $S(R)$, only with R shrunken down to a hypersurface Σ , and with $\mathfrak{A}_\Sigma$ then being the algebra generated (formally) by the "initial data" operators $\phi(x)$ and $\dot{\phi}(x)$ for $x \in \Sigma$.

The point now is that when Σ is a Cauchy surface for the region R , we have $\mathfrak{A}_\Sigma = \mathfrak{A}_R$, and therefore $S(\Sigma)$ will coincide with the more globally defined entropy $S(R)$. By taking for R the so-called *domain of dependence* of Σ , we can thus express the entropy of any desired hypersurface as the entropy of a spacetime region. And this in turn will let us express it directly in terms of the correlation function $\langle \phi(x)\phi(y) \rangle$.

In application to a black hole spacetime, Σ would be the exterior portion of the hypersurface for which we desired the entanglement entropy, and the region R would then be given by

$$R = \text{future}(\Sigma) \cap \text{past}(H) = D^+(\Sigma) \qquad (1)$$

† For $\dim \mathfrak{H} < \infty$ this is proven in [6].

Remark Time-reversing the above, we could also have taken $R = D^-(\Sigma)$, but the analog of (1) would not hold with that choice, making it apparently less convenient when transposed to a causet.

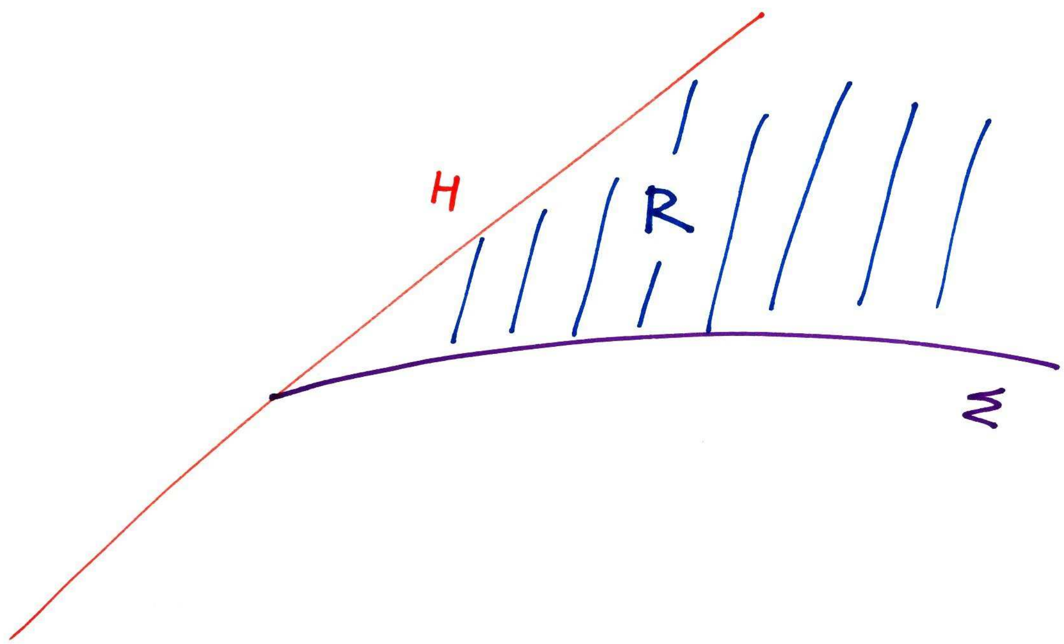

Figure 2. The entropy of the surface Σ can be identified with the entropy of the region R

In summary, *entanglement entropy is a special case of the entropy $S(R)$ of a spacetime region.* Our task now is to find a simple formula for $S(R)$, assuming, as always, that ϕ is free and "gaussian".

Sketch of a derivation of a formula

What follows is in parts still a "work in progress"; not every loose end has been tied up.

We have arrived at an algebra $\mathfrak{A} = \mathfrak{A}(R)$ and an expectation-functional or "state" $\langle \cdot \rangle$ thereon. In order to derive from these structures an entropy $S(R)$, we will seek to represent $\mathfrak{A}(R)$ irreducibly in a hilbert space $\mathfrak{H} = \mathfrak{H}(R)$ and to find therein a density-matrix $\rho : \mathfrak{H} \to \mathfrak{H}$ such that $\langle A \rangle = \mathrm{Tr}\, \rho A$ for all $A \in \mathfrak{A}$. In what follows I will omit the reference to R, as if R were the full spacetime or causet as the case may be. There is no loss of generality in doing so, since we will not need to refer again to spacetime points (respectively causet elements) outside of R. Also I will write "ϕ^j" for an individual field value. This notation is more convenient than "$\phi(x)$" and it is also more apt for a causet C , where the index j runs literally from 1 to N , $N = |C|$ being the cardinality of C.

Remark The condition that $\mathfrak{A}$ act irreducibly in $\mathfrak{H}$ is crucial here. If we dropped it, we could, by "purification", always find a representation in which $\operatorname{Tr} \rho \log \rho$ would vanish because ρ would be the projector onto a single vector in $\mathfrak{H}$. This would be the case in the so-called GNS representation, and for that matter, in the representation of the ϕ^j in the original, global hilbert space, assuming the original global state to have been pure. In view of these observations, it's somewhat disconcerting that, strictly speaking, the existence of an irreducible representation is not always guaranteed. Consider for example the "region" of a causet consisting of a single element e. The corresponding subalgebra $\mathfrak{A}$ will be generated by a single field-value $\phi(e) =: q$. Now by assumption, $\langle q \rangle = 0$, but in general $\langle qq \rangle$ will not vanish. If it does not, then q cannot act irreducibly in any representation (because then Schur's lemma $\Rightarrow q = c \in \mathbb{R} \Rightarrow q = \langle q \rangle = 0 \Rightarrow \langle qq \rangle = \langle 0 \rangle = 0$). In such a case, the entropy could not be defined by the means we have adopted. However, the "strictly algebraic" definition we remarked on earlier would still seem to apply and to yield $S = \infty$. In the following, we will simply assume that an irreducible representation does exist. The formula (9) which we will derive will not actually depend on this assumption if interpreted as in (8). It would however yield $S = 0$ in our one-element example, rather than $S = \infty$.

Because we are dealing with a free field, the commutator of ϕ^j with ϕ^k is a c-number and we can write

$$[\phi^j, \phi^k] = i\Delta^{jk} \tag{2}$$

for some real, skew matrix Δ^{jk}. We define similarly the *Wightman function*[♭] as the hermitian matrix

$$W^{jk} \equiv \langle \phi^j \phi^k \rangle \ .$$

[♭] Some connections between the Wightman function and the entropy of a Gaussian density matrix are also discussed in [7].

Notice that Δ is twice the imaginary part of W .

Assume now that our theory is "Gaussian" (or "Wickian") in the sense that the bosonic form of *Wick's rule* holds with $\langle \phi \rangle = 0$:

$$\langle \phi\phi \cdots \phi \rangle = \sum \langle \phi\phi \rangle \cdots \langle \phi\phi \rangle$$

(The sum is taken over all ways of pairing the ϕ's, and within each pairing the original odering must be preserved.) Thanks to this assumption the "two-point function" W^{jk} determines the theory fully.

For example its imaginary part determines the "equations of motion" for ϕ. Thus (thinking for a moment of the theory on the entire spacetime or causet), if the sequence α_j is in the kernel of Δ in the sense that $\sum \Delta^{kj}\alpha_j = 0$ then $A = \sum \alpha_j \phi^j$ vanishes as well. (proof: When $\alpha \in \ker \Delta$, A commutes with every ϕ^j, as follows from the calculation $[\phi^j, A] = [\phi^j, \alpha_k \phi^k] = i\Delta^{jk}\alpha_k = 0$. But since the ϕ^j act irreducibly in $\mathfrak{H}$, this implies that $A = c\mathbf{1}$, and then $c = \langle A \rangle = \alpha_j \langle \phi^j \rangle = 0$ whence $A = 0$.) This last fact yields a set of linear relations among the operators ϕ. In the continuum they are just the equations of motion for $\phi(x)$ (in our case the Klein-Gordon equation), but in the causet they yield only relatively few conditions because the *exact* kernel of Δ is typically rather small.

Now our algebra $\mathfrak{A}$ is generated by the individual field-variables ϕ^j, and we can therefore characterize it fully by specifying the relations among these generators. But since we have taken our field to be non-interacting, this is quite simple to do. In addition to the linear relations just described there are only the bilinear relations (2) (the "canonical commutation relations"). Moreover, we have just seen that any zero-eigenvector of the commutator-form Δ^{ij} results in a linear dependence among the ϕ^j . Hence, by passing to a linearly independent set of generators, we can assume that Δ^{ij} is invertible.

It follows that there exists a basis for the ϕ^j of the form q^α, p_α ($\alpha = 1 \cdots n$) such that

$$[q^\alpha, p_\beta] = i\delta^\alpha_\beta$$

By construction this basis block-diagonalizes Δ, meaning it block-diagonalizes the skew part of W . (Recall that $W^{jk} - W^{kj} = i\Delta^{jk} = [\phi^j, \phi^k]$.) I claim one can also choose it to diagonalize the symmetric part,

$$R^{jk} \equiv \mathrm{Re}\, W^{jk} = \frac{1}{2}\langle \{\phi^j, \phi^k\} \rangle \ ,$$

hence to block-diagonalize W^{jk} itself.⋆ With this, our problem splits up into a product of individual problems, each concerning a single pair (q, p) — a single degree of freedom — that you might imagine as position and momentum for a free particle or harmonic oscillator (or as the field-components ϕ^j for a 2-element causet, a 2-chain). It thus suffices to evaluate the entropy in that special case.

the entropy for a single degree of freedom

Our task is now as follows. Given a conjugate pair of variables q and p such that $[q, p] = i$, and given the correlators

$$\langle qq \rangle \quad \langle pp \rangle \quad \mathrm{Re}\,\langle qp \rangle$$

for a Gaussian density-matrix ρ, to find $S(\rho) = \mathrm{Tr}\,\rho \log \rho^{-1}$. That ρ is Gaussian means that in a q-basis it takes the form

$$\rho(q, q') \equiv \langle q|\rho|q' \rangle = (cst) \exp\left(-\frac{A}{2}(q^2 + q'^2) + \frac{iB}{2}(q^2 - q'^2) - \frac{C}{2}(q - q')^2\right) \tag{3}$$

Then $S(\rho)$ must be some function of the parameters A, B, C, and our task is to determine this function.

To that end, notice first that S must be dimensionless and invariant under unitary transformations. From this it can be shown that S can depend on the correlators only in the combination $\langle qq \rangle \langle pp \rangle - (\mathrm{Re}\,\langle qp \rangle)^2 = \det R / \det \Delta$, where in this simple 2×2 situation, Δ and R reduce to

$$\Delta = 2\,\mathrm{Imag}\begin{pmatrix} \langle qq \rangle & \langle qp \rangle \\ \langle pq \rangle & \langle pp \rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$R = \mathrm{Re}\begin{pmatrix} \langle qq \rangle & \langle qp \rangle \\ \langle pq \rangle & \langle pp \rangle \end{pmatrix} = \begin{pmatrix} \langle qq \rangle & \mathrm{Re}\,\langle qp \rangle \\ \mathrm{Re}\,\langle qp \rangle & \langle pp \rangle \end{pmatrix}$$

⋆ In fact, diagonalization is possible in general, without assuming that the ϕ^j are linearly independent. Given any positive-semidefinite matrix W, with associated real matrices R and Δ such that $W = R + i\Delta/2$, one can introduce a (real) basis in which R is diagonal and Δ is block-diagonal with 2×2 blocks. Moreover one can arrange that the non-zero matrix elements $R^{jj} = 1$. The proof is slightly too long to reproduce here, but see [8] for a similar result that applies when Δ is invertible.

It turns out that (as a short calculation will confirm)

$$\langle qq \rangle \langle pp \rangle - (\mathrm{Re}\,\langle qp \rangle)^2 = C/2A + 1/4$$

whence the entropy depends only on the ratio C/A, while B drops out entirely. But with B set to 0, we can take over from [9] the entropy calculation there, with the result [†]

$$-S = \frac{\mu \log \mu + (1-\mu)\log(1-\mu)}{1-\mu} \ ,$$

where

$$\mu = \frac{\sqrt{1+2C/A} - 1}{\sqrt{1+2C/A} + 1}$$

Putting the pieces together yields now

$$S = (\sigma + 1/2)\log(\sigma + 1/2) - (\sigma - 1/2)\log(\sigma - 1/2) \tag{4}$$

where I have defined the eigenvalues of $\Delta^{-1}R$ (which are purely imaginary) to be $\pm i\sigma$: spectrum$(\Delta^{-1}R) = \pm i\sigma$.

This nicely symmetrical expression is already quite simple, but we can simplify it still further by swapping the eigenvalues of $\Delta^{-1}R$ for those of $\Delta^{-1}W = \Delta^{-1}R + i/2$. (Recall that $W = R + i\Delta/2$.) Writing these latter eigenvalues as $\pm i\omega_\pm = i(\frac{1}{2} \pm \sigma)$, brings S finally to the convenient form

$$S = \omega_+ \log \omega_+ - \omega_- \log \omega_- \tag{5}$$

Notice in this connection that positivity of the correlation matrix (also known as the uncertainty principle) implies that $\sigma \geq 1/2$, whence ω_+ and $\omega_- = \omega_+ - 1$ are both ≥ 0.

the entropy in full

The formula (5) belongs to a single degree of freedom, corresponding to one block of our block-diagonalized matrix W. Summing (5) over all the blocks then yields the total entropy

[†] Might there be a more direct route from (3) to $S(\rho)$, perhaps via a diagonalization of ρ or via a use of the "replica trick"?

in the form of a sum over the spectrum of the full matrix $\Delta^{-1}W$, which for short I will call L, or more conveniently iL in oder that its eigenvalues be real:

$$\Delta^{-1} W = iL \tag{6}$$

Denoting the eigenvalues of L by λ, we have then

$$S = \sum \lambda \, \log |\lambda| \tag{7}$$

We have seen that the eigenvalues λ are all real, even though L is in general neither real nor symmetric. We have also seen that each negative λ is paired with a positive eigenvalue $1 - \lambda$.

As derived, this last formula relies on the fact that the q^α and p^α are all linearly independent. When we revert to the original matrices W^{jk} and Δ^{jk}, we will encounter (after their diagonalization) a block submatrix consisting entirely of zeroes. In this subspace, Δ is obviously not invertible, but neither is there any contribution to S. To adapt our formula (7) to this circumstance, we can rewrite the eigenvalue equation for $\Delta^{-1}W$ so that it ignores this block of zeroes. To wit, we define the eigenvalues λ as the solutions of the equation $W^{jk}v_k = i\lambda\Delta^{jk}v_k$, where it is understood that $\Delta^{jk}v_k$ must not vanish (just as v must not vanish in the more traditional eigenvalue equation $Lv = \lambda v$). In other words, we define the eigenvalues of L as the solutions of the equation

$$Wv = i\lambda\Delta v \qquad\qquad (\Delta v \neq 0) \ . \tag{8}$$

With this understanding (and with the usual convention that $0 \log 0 = 0$), we have finally

$$S = \ \mathrm{Tr} \ L \ \log |L| \equiv \sum_{\lambda} \lambda \ \log |\lambda| \tag{9}$$

A remarkably simple formula!

Remark Derived under the assumption that the ϕ^j can be represented irreducibly in some Hilbert space, equations (8) and (9) give a good account of the entropy when the kernels of Δ and R (or equivalently W) coincide. From the condition $W \geq 0$ that W is positive semidefinite, it follows that $\ker \Delta \supseteq \ker R$, but the converse inclusion is not guaranteed in general. When it fails, no irreducible representation exists (as seen in an earlier remark), and our entropy is to that extent ill-defined. If we adopt the more general "strictly algebraic" definition described earlier, the resulting entropy will diverge. By way of illustration, consider our earlier example of a single "field operator" $\phi^1 = q$ with $\langle q \rangle = 0$ and $\langle qq \rangle = s^2$ (and with $\mathfrak{A}$ being the algebra of polynomials in q). One can think of q in this case as a classical random variable with gaussian distribution function $1/\sqrt{2\pi s^2}\exp(-x^2/2s^2)$, the pure states corresponding to specific sharp values $q = x$ for $x \in \mathbb{R}$. Since these states are uncountably infinite in number, our "strictly algebraic" entropy will be infinite, absent some sort of short-distance cutoff or regulator.

Remark Given the commutator function Δ^{jk}, the construction of [10] and [11] produces a distinguished "vacuum" by taking the two-point function W to be the positive projection of $i\Delta$, i.e. by setting $W = \sum'_n w_n |n\rangle\langle n|$, where the sum is over the postive eigenvalues w_n of the matrix $i\Delta = \sum_n w_n |n\rangle\langle n|$. Evidently, the positive eigenvalues produce solutions of (8) with $\lambda = 1$, while the negative eigenvalues produce solutions with $\lambda = 0$. The net entropy therefore vanishes, as it must since the vacuum in question is the "ground state" of a Fock-type representation, hence a pure state.

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